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Design of PID controllers using semi-infinite programming

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Abstract

The PID controller is widely used, and several methods have been proposed for choosing the controller parameters to achieve good performance. The controller tuning problem is set up as a semi-infinite program (SIP), with the integrated squared error (ISE) or the H_∞ norm of the frequency domain error function ($|E(s)|_\infty$) as the objective function, and H_∞ constraints for robustness and noise attenuation. Previous authors considered discrete points to enforce the H_∞ constraints, however this is an outer approximation that does not guarantee a feasible point. When a feasible point can be found, it may require multiple iterations with a finer and finer discretisation. Here, the SIP is solved using a global optimisation algorithm. Several numerical experiments show that the proposed formulation converges quickly (<10 seconds) and gives sensible controller tuning values without the need to apply expert knowledge to the tuning problem. These results suggest that this is an attractive method for automated controller tuning.

Keywords: Controller tuning; Global Optimisation; Process Dynamics and Control; Semi-infinite Programming

1. Introduction

The PID controller has found widespread use in industry and there are many methods in the literature to tune PID parameters. Typically, tuning involves a trade-off between rejecting disturbances and robustness to uncertainty (Åström and Hägglund, 2006). Finding parameters by trial and error is time-intensive, which has led to the formulation of tuning rules, e.g. the Ziegler-Nichols tuning rule and SIMC, see Åström and Hägglund (2006) for an overview. An alternative to tuning rules, is to find controller parameters by solving an optimisation problem. Optimisation-based tuning is a powerful tool, especially when system complexity, non-standard parameterisations, or requirements on performance and robustness mean that tuning rules are ill-suited (Grimholt and Skogestad, 2018; Åström and Hägglund, 2006).

Balchen (1958) presented the first “modern” formulation of the PID optimisation problem, that explicitly included a performance and robustness trade off. Since then, various authors have proposed different formulations, see e.g. Soltesz et al. (2017). Here, we place constraints on the H_∞ norm of transfer functions, i.e. the constraints should be satisfied for all considered frequencies ($w \in \Omega \subset \mathbb{R}_+$), which means there are an infinite number of constraints (Grimholt, and Skogestad, 2018; Soltesz et al. 2017).

Previous authors (Grimholt, and Skogestad, 2018; Soltesz et al. 2017) discretised the frequencies to form a finite problem, e.g., Grimholt, and Skogestad (2018) used 10 000 points. This is an outer approximation that does not guarantee a feasible point. It also raises the problem of how to select the discretisation frequencies. If we consider the PID tuning problem as one in which the constraints must be satisfied, then this means that multiple iterations with a finer discretisation or the use of expert knowledge to choose a good prior discretisation may be necessary.

In this work we use the global optimisation algorithm proposed by Djelassi and Mitsos (2017) to solve the semi-infinite PID tuning problem. This algorithm iteratively solves discretised subproblems, where at each iteration a new discretisation point is added at the frequency that results in the largest constraint violation at the incumbent solution. To facilitate the global optimisation algorithm, we use an objective function in the frequency domain. Initial results show that the proposed formulation converges in reasonably quick computation times (<10 seconds) and gives sensible controller tuning values without the need to apply expert knowledge to the tuning problem.

1.1. System

We consider the closed loop linear system in Figure 1, with disturbances at the plant input and output (d_u and d_y), and noise (n) entering the system at the measurement output. The system is represented by the following transfer functions (Åström and Hägglund, 2006):

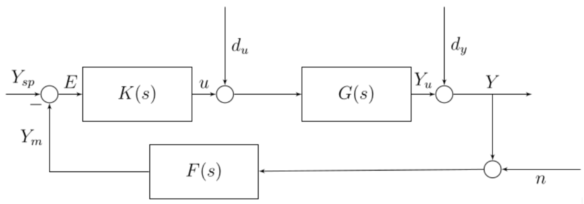


Figure 1. Block diagram of closed loop system. K(s) is the controller, G(s) is the process and F(s) is the filter.

$$S(s) = \frac{1}{1 + G(s)K(s)}, \qquad T(s) = 1 - S(s), \qquad TF(s) = T(s)F(s),$$

$$GS(s) = G(s)S(s), \qquad KS(s) = K(s)S(s), \qquad KFS(s) = K(s)F(s)S(s),$$

where s is the complex frequency ($s = iw$), and $S(s)$ and $T(s)$ are the sensitivity and complementary sensitivity functions, respectively. Here, we consider the case of pure error feedback ($F = 1$). The controller error, E, is the difference between the measured output (y) and setpoint (y_s):

$$-E(s) = y - y_s = S(s)d_y + GS(s)d_u - T(s)n. \tag{1}$$

In this work we consider PID controllers that are parameterised in the linear form:

$$K(s) = k_p + \frac{k_i}{s} + k_d s, \tag{2}$$

where k_p, k_i, and k_d are the tuning parameters. In this form the optimiser can selected a PID subtype, e.g. setting k_d to zero yields a PI controller.

1.2. Objective

We wish to pick control parameters that minimise the error after some disturbance. Various performance indices have been proposed, with the most widely used measure being the integral absolute error (IAE):

$$IAE = \int_0^\infty |e(t)|\, dt. \tag{3}$$

This formulation requires the error function in the time domain ($e(t)$). Finding the time domain error function generally involves explicit simulation or taking the inverse Laplace transform. Balchen (1958) proposed the use of a performance index in the frequency domain that approximates the IAE. The rationale behind the approximation is that $|e(t)| = e(t)\frac{|e(t)|}{e(t)}$, where if $e(t)$ is oscillatory then the fraction defines a square wave. The IAE can then be approximated by introducing a sine wave with free parameters w and a, that are chosen to maximise the integral, i.e. reduce the approximation error. This allows one to write the objective in the frequency domain:

$$IAE = \int_0^\infty |e(t)|\, dt \approx \max_{\mathrm{a,w}} \int_0^\infty e(t) \sin(wt + a)\, dt \tag{4}$$

$$= \max_w |E(iw)| = |E(s)|_\infty = HIE,$$

where $|\cdot|_\infty$ is the H_∞ norm. For convenience, we shall refer to this as the H-infinity error (HIE). The HIE is bounded by the integral error (IE) and IAE: $IE \leq HIE \leq IAE$. If the system is well-dampened, then $IE \approx HIE \approx IAE$. Using Parseval's theorem, the integral squared error can be (exactly) represented in the frequency domain:

$$ISE = \int_0^\infty e(t)^2\, dt = \frac{1}{\pi}\int_0^\infty |E(iw)|^2 dw. \tag{5}$$

1.3. Robustness

We enforce robustness by constraining the maximums in the sensitivity and complementary sensitivity functions M_S and M_T, where

$$M_S = |S(iw)|_\infty, \qquad M_T = |T(iw)|_\infty.$$

The magnitude of M_S and M_T, describe the sensitivity of the system to process uncertainty or change, e.g., M_S gives the worst-case amplification of a disturbance and, on a Nyquist plot, is the distance from the loop transfer function to the point (-1,0).

Constraining the magnitude of M_S and M_T defines circles on the Nyquist plot that the loop transfer function must lie out of. A combined sensitivity constraint can be defined that covers both excluded regions. For $M = M_S = M_T$, this constraint is a circle on the Nyquist plot with centre (C, 0) and radius R given by (Åström and Hägglund, 2006):

$$C = -\frac{2M^2 - 2M + 1}{2M^2 - 2M}, \qquad R = -\frac{2M - 1}{2M^2 - 2M}.$$

1.4. Noise attenuation

It is also desirable to limit control usage due to noise. This can be performed by bounding the noise amplification ratio, $\frac{\sigma_u^2}{\sigma_n^2}$, where σ_u^2 and σ_n^2 are the variances of the control and noise respectively. Let $\phi_n(w)$ be the unknown spectral density of the (unclassified) noise, and Q be the transfer function from noise to the control signal ($Q = -KFS$, see Figure 1). The following inequality holds (Soltesz, et al. 2017):

$$\sigma_u^2 \leq |Q|_\infty^2 \sigma_n^2. \tag{6}$$

Thus, the constraint $|Q|_\infty \leq M_Q$ conservatively constrains the noise amplification ratio. This inequality can be written in the form:

$$|KF(iw)| - M_Q|1 + L(iw)| \leq 0, \qquad \forall w \in \Omega \subset \mathbb{R}_+, \tag{7}$$

where Ω defines the range of frequencies considered.

1.5. Optimisation problem

Semi-infinite programs are optimisation programs with a finite number of variables, and an infinite number of constraints. In the PID problem we have an infinite number of constraints as the constraint must hold for all considered frequencies ($w \in \Omega \subset \mathbb{R}_+$). The optimisation problem for some performance index (P_I) in the frequency domain is:

$$\min_{k_p, k_i, k_d} \eta \tag{8.a}$$

$$P_I(iw) - \eta \leq 0, \qquad \forall w \in \Omega \subset \mathbb{R}_+, \tag{8.b}$$

$$R^2 - |C - L(iw)| \leq 0, \qquad \forall w \in \Omega \subset \mathbb{R}_+, \tag{8.c}$$

$$|KF(iw)| - M_Q|1 + L(iw)| \leq 0, \qquad \forall w \in \Omega \subset \mathbb{R}_+, \tag{8.d}$$

where the constraints are explicitly parameterised by the frequency.

2. Numerical examples

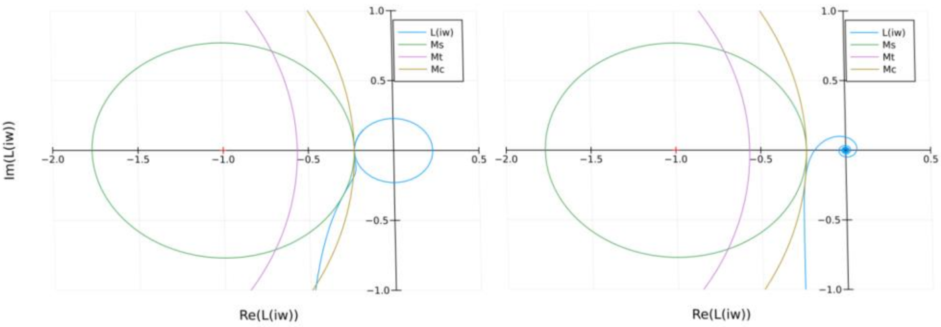
This work is coded in Julia and with the use of the global optimisation package EAGO.jl (Wilhelm and Stuber, 2020), GLPK (Makhorin, 2008), IPOPT (Wächter and Biegler, 2006), and the JuMP modelling language (Dunning, et al. 2017).

2.1. First order process with time delay

Consider the system from Grimholt and Skogestad (2018) with transfer functions:

$$G(s) = \frac{\exp(-s)}{s + 1} \qquad\qquad F(s) = \frac{1}{0.001s + 1}$$

To compare with the published results, we use the same weighted cost of the error from a step disturbance in u and y: $\eta = \frac{1}{1.56} HIE_{dy} + \frac{1}{1.42} HIE_{du}$. We enforce constraints on the sensitivity and complementary sensitivity with $M_S = M_T = 1.3$ and only consider frequencies w in the interval [0.01 100]. No constraint is used for the input usage.



(a) (b)

Figure 2: Nyquist plots of first order process with time delay. Left plot has constraints on maximum sensitivity and complementary sensitivity. Right plot has constraints on combined sensitivity and noise attenuation.

The optimiser finds the parameters $[0.51, 0.54, 0.23]$ in 2.6 seconds, with the Nyquist plot shown in Figure 2a. This closely matches the reported solution of $[0.52, 0.53, 0.22]$, despite the use of HIE instead of the IAE (Grimholt, and Skogestad, 2018).

For comparison, introducing a constraint on input usage ($M_Q = 1.0$) and using the combined circle constraint gives the control parameters [0.32 0.28 and 0.01], with the Nyquist plot shown in Figure 2b.

2.2. Third order process with inverse response

Consider the system process transfer functions:

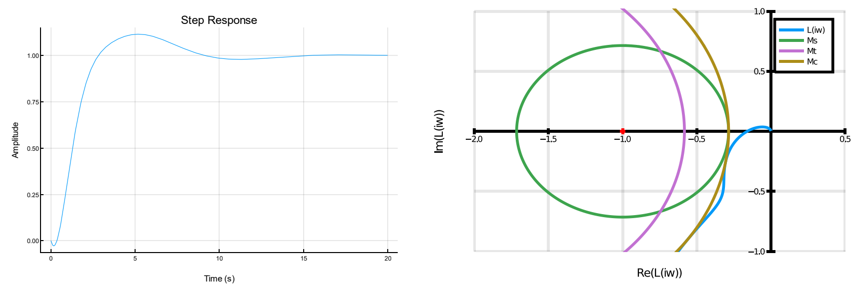
$$G(s) = \frac{1 - 0.2s}{(s+1)^3}, \qquad F(s) = 1.$$

We consider a constraint on the maximum combined sensitivity (≤ 1.3) and error function $E(s) = GS(s)d_u$. We consider frequencies in the interval $[0.01\ 100]$, and bounds on controller parameters of 0.0 and 2.0.

The optimisation is performed with HIE and ISE as the objective, giving parameters of [1.58, 1.00, 1.73] and [1.54, 1.05, 1.87] respectively, in less than 5 seconds each. The system response using the HIE parameters is shown in Figure 3.

2.3. Discussion

Despite the potential for HIE to go to zero, this did not occur in the above examples. Numerical experiments have shown that this generally occurs with oscillatory systems or large upper bounds on the control parameters and no constraint on input usage. Providing good bounds on the control parameters (e.g. by using a tuning rule) can improve the speed of optimisation. If the bounds could ensure that the control system is well-dampened, then $HIE \approx IAE$. The proposed SIP formulation can be readily extended to other linear fixed-order controllers.



(a) (b)

Figure 3: Step response and Nyquist plot for third order process with inverse response. HIE is used as the objective with no constraint on input usage.

3. Conclusions

We demonstrate that the robust PID tuning problem can be formulated and solved as a semi-infinite program, entirely in the frequency domain, using the HIE or ISE as objective functions. Robustness is enforced via H_∞ constraints on the sensitivity and complementary sensitivity functions, or an H_∞ constraint on the combined sensitivity. Control usage is restricted via an H_∞ constraint on the noise amplification ratio. On a range of systems, sensible controller parameters were found, typically in less than 10 seconds. Potential further work could include an extension to multiple output systems, or other controllers.

4. Acknowledgments

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